Amazing quadratic equations-1

Meijun Zhu

6-14-2025

We will show how solving quadratic equations inspires us to expand number fields.

Using the basic properties of equality, we can derive the following rule: For any two number x and y,

$$x^2 = y^2 \ \ is\ equivalent\ to\ x = y\ or\ x = -y.$$

This result essentially follows from the fundamental rule:

$$xy = 0 \ \ is\ equivalent\ to\ x = 0\ or\ y = 0.$$

Setting aside some technical details (in particular, the method of completing the square, which all students are expected to know), quadratic equations typically fall into three categories:

1. **Equations with only rational solutions.**
2. **Equations with irrational but no rational solutions,** requiring us to expand the number system from the rational numbers $\mathbb{Q}$ to the real numbers $\mathbb{R}$.
3. **Equations with no real solutions but complex solutions,** requiring a further expansion from $\mathbb{R}$ to the complex numbers $\mathbb{C}$.

Type 1. Rational solutions.

Example 1: Solving the equation

$$x^2 = 1.$$

Solution: Using the rule above:

$$x^2 = 1 \ \Leftrightarrow\ x^2 = 1^2 \ \Leftrightarrow x = 1\ or\ x = -1,$$

We conclude that $x = 1, or\ x = -1$.

Type 2. Irrational solutions.

Example 2: Solve the equation

$$x^2 = 2.$$

Before we learn how to solve this equation, we must first ask: *Are there any numbers whose square is 2?*

Mathematically, we can show that no integer or rational number (fraction) satisfies this condition. Therefore, to solve the equation, we must expand our number system by introducing a new number: the **positive square root of 2**, denoted $\sqrt{2}$, which satisfies

$$\left(\sqrt{2}\right)^2 = 2.$$

Now we can solve the equation.

Solution to Example 2:

$$x^2 = 2 \iff x^2 = \left(\sqrt{2}\right)^2 \iff x = \sqrt{2}\ or\ x = -\sqrt{2}.$$

Since $\sqrt{2}$ is not a fraction, we call it an irrational number. Solving this type of quadratic equations requires us to expand the number system from the rational numbers $\mathbb{Q}$ to the real numbers $\mathbb{R}$.

Type 3. Complex solutions.

Example 3: Solve the equation

$$x^2 = -1.$$

Are there any real number whose square is -1?

By the properties of inequality, we know that there is no real number whose square is -1.

However, this limitation inspires us to **introduce** a new symbol, called the **imaginary unit (or imaginary symbol)**, denoted by i, with the defining property:

$$i^2 = -1.$$

Now we can solve the equation.

Solution to Example 3:

$$x^2 = -1 \iff x^2 = i^2 \iff x = i \ \ or \ x = -i.$$

In the same way, we can solve similar equations.

Example 4: Solve the equation

$$x^2 = -4.$$

Solution:

$$x^2 = -4 \iff x^2 = (2i)^2 \iff x = 2i \ \ or \ x = -2i.$$

Any number of the form bi, where b is a real number, is called an **imaginary number**.

So far, so good. But what happens in the next example?

Example 5: Solve the equation

$$x^2 = i.$$

We observe that there is no real number or purely imaginary number whose square is i.

This leads us to **introduce** a broader class of numbers called **complex numbers**, of the form

$$a + bi,$$

where a, b are real numbers.

As agreed at the beginning of our study of algebra, complex numbers must satisfy the **three fundamental laws of algebra**:

- the **commutative law**,
- the **associative law**, and
- the **distributive law**.

Using these laws, we can verify that

$$\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i\right)^2 = i.$$

Thus, we can write the solution as:

Solution to Example 5:

$$x^2 = i \Leftrightarrow \; x^2 = \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i\right)^2 \;\Leftrightarrow\; x = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i \; or \; x = -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\,i.$$

But how can we systematically find such solutions?

Let us consider the general case: For any complex number $a+bi$, we seek a complex number $x+yi$ (where x, y are real numbers), such that

$$(x+y\,i)^2 = a+bi.$$

Expanding the left-hand side and equating real and imaginary parts, we get:

- Real part: $x^2 - y^2 = a$
- Imaginary part: $2xy = b$

To find x and y, we can also use the **modulus** of both sides:

$$|x+yi|^2 = x^2+y^2 = |a+bi| = \sqrt{a^2+b^2}.$$

From these, we derive:

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}, \quad y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}.$$

Since $2xy = b$, we can determine the signs of x and y accordingly (they have the same or opposite signs depending on the sign of b).

Example 6: Solve the equation for z:

$$z^2 = 1 + i.$$

Solution: We seek a complex number $x + yi$ (where x, y are real numbers), such that

$$(x + y\,i)^2 = 1 + i.$$

Expanding the left-hand side and equating real and imaginary parts, we get:

- Real part: $x^2 - y^2 = 1$
- Imaginary part: $2xy = 1$

To find x and y, we can also use the **modulus** of both sides:

$$|x + yi|^2 = x^2 + y^2 = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

From these, we derive:

$$x^2 = \frac{1+\sqrt{2}}{2}, \quad y^2 = \frac{-1+\sqrt{2}}{2}.$$

Since $2xy = 1$, the signs of x and y are the same, we conclude:

$$\begin{cases} x = \sqrt{\dfrac{1+\sqrt{2}}{2}} \\ y = \sqrt{\dfrac{-1+\sqrt{2}}{2}} \end{cases}, \qquad or \qquad \begin{cases} x = -\sqrt{\dfrac{1+\sqrt{2}}{2}} \\ y = -\sqrt{\dfrac{-1+\sqrt{2}}{2}} \end{cases}.$$

Thus, we obtain two solutions

$$Z = \sqrt{\frac{1+\sqrt{2}}{2}} + \sqrt{\frac{-1+\sqrt{2}}{2}} \cdot i, \quad or \quad Z = -\sqrt{\frac{1+\sqrt{2}}{2}} - \sqrt{\frac{-1+\sqrt{2}}{2}} \cdot i \ .$$

Summary

By solving various types of quadratic equations, we have extended our number system step by step:

- from **rational numbers** $\mathbb{Q}$,
- to **real numbers** $\mathbb{R}$,
- and ultimately to **complex numbers** $\mathbb{C}$.

Moreover, we have shown that for any complex number $a + bi$, the equation

$$(x + y\,i)^2 = a + bi$$

always has two complex solutions. This is a special case of the **Fundamental Theorem of Algebra** for quadratic (second-degree) polynomial equations.